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Evolution of Special Subsets of $\mathbb{C}^2$

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0. INTRODUCTION

Let K be a compact subset of $\mathbb{C}^2$. In [ST_2, ST_3] we gave the notion of evolution of K by Levi form. If

$$L(u) = (\delta_{\alpha\beta} - |\partial u|^{-2}\, u_{\bar{\alpha}} u_{\beta})\, u_{\alpha\bar{\beta}}$$

denotes the Levi operator and K is the zero set of a continuous function g, this amounts to studying the parabolic problem

$$\begin{cases} u_t = L(u) & \text{in} \quad \mathbb{C}^2 \times (0, +\infty) \\ u = g & \text{on} \quad \mathbb{C}^2 \times \{0\} \end{cases} \qquad (\bigstar)$$

(here $u_\alpha = \partial u/\partial z_\alpha$, $\alpha = 1, 2$, and z_1, z_2 are complex coordinates in $\mathbb{C}^2$). We proved that ($\bigstar$) has a unique weak (viscosity) solution $u = u(x, t)$, $x = (z_1, z_2)$, which is constant for $|x| + t \gg 0$ and that the family $\{\mathscr{E}_t^{\mathscr{L}}(K)\}_{t \geqslant 0}$, $\mathscr{E}_t^{\mathscr{L}}(K) = \{x \in \mathbb{C}^2 : u(x, t) = 0\}$ is independent on the chosen g. $\{\mathscr{E}_t^{\mathscr{L}}(K)\}_{t \geqslant 0}$ describes the evolution of K.

Focusing our attention on the effects of pseudoconvexity on evolution we proved that if Ω is a bounded pseudoconvex domain of $\mathbb{C}^2$ with boundary of class C^3, then the evolution $\{\bar{\Omega}_t\}_{t \geqslant 0}$ of $\bar{\Omega}$ is contained in $\bar{\Omega}$.

We also conjectured that if Ω is not pseudoconvex then $\bar{\Omega}_t \nsubseteq \Omega$ for some $t > 0$. This is actually true.

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THEOREM 0.1. *Let $\Omega \subset \mathbb{C}^2$ be a bounded domain. Assume that Ω is not pseudoconvex. Then there is $t^\circ > 0$ such that for every $0 < t < t^\circ$, $\Gamma_t = (b\Omega)_t$ is not contained in $\bar{\Omega}$. More precisely, if $b\Omega$ is not pseudoconvex at z^*, in the sense that $B(z^*, \varepsilon) \cap \Omega$ is not pseudoconvex for any $\varepsilon > 0$ ($B(z^*, \varepsilon)$ the ball of radius ε centered at z^*), then for every $\varepsilon > 0$ there is a $t^\varepsilon > 0$ such that $B(z^*, \varepsilon) \cap \Gamma_t$ is not contained in $\bar{\Omega}$ for $0 < t < t^\varepsilon$.*

Proof. If Ω is not pseudoconvex at $z^* \in b\Omega$ in the sense formulated above then for every $\varepsilon^\circ > 0$ there is a point $z^\circ = z(\varepsilon^\circ) \in B(z^*, \varepsilon^\circ) \cap b\Omega$ and a smooth domain $\Omega' = \Omega(\varepsilon^\circ)$ such that $\Omega' \subset \Omega$, $\bar{\Omega}' \subset \Omega \cup \{z^\circ\}$ and $\Omega' \cap B(z^\circ, r)$ is strictly pseudoconcave at the points of $b\Omega' \cap B(z^\circ, r)$ for some $r > 0$. We can assume that there is a defining function v for $b\Omega'$ with the following properties:

(1) $v \in C^\infty(\mathbb{C}^2)$, $v(z) = 2$ for $|z| \gg 0$, $dv(\zeta) \neq 0$ for $\zeta \in b\Omega'$, $L(v)(\zeta) < 0$ for $\zeta \in B(z^\circ, r)$, and

$$\Omega' = \{z \in \mathbb{C}^2 : v(z) < 0\}.$$

Consider now any "weakly defining" function g_1 for $b\Omega$, i.e., $g_1 \in C^0(\mathbb{C}^2)$, $g_1(z) = 3/2$ for $|z| \gg 0$ and

$$\Omega = \{z \in \mathbb{C}^2 : g_1(z) < 0\}.$$

Clearly $g_1(z^\circ) = 0$, $g_1(z) < 0$ for $z \in b\Omega' \setminus \{z^\circ\}$.

Let $g_2 = \min(v, g_1)$. One has

$$\Omega = \{z \in \mathbb{C}^2 : g_2(z) < 0\},$$

since $\Omega' \subset \Omega$, and $g_2(z) = 3/2$ for $|z| \gg 0$.

Finally let

$$g(z) = \begin{cases} 2g_2(z)/3 & \text{if } g_2(z) \geqslant 0 \\ 2g_2(z) & \text{if } g_2(z) < 0. \end{cases}$$

It is clear that g has the following properties:

(2) $g \in C^0(\mathbb{C}^2)$, $g(z) = 1$ for $|z| \gg 0$, $g(z^\circ) = v(z^\circ) = 0$, $g(z) < v(z)$ for $z \in \mathbb{C}^2 \setminus \{z^\circ\}$, and

$$\Omega = \{z \in \mathbb{C}^2 : g(z) < 0\}.$$

(Regarding the last inequality, note that if $g_2(z) < 0$, $z \neq z^\circ$ then $z \in b\Omega' \setminus \{z^\circ\}$, which set is disjoint from $\bar{\Omega}'$, hence $v(z) > 0$.) Now let u be the weak solution of the parabolic problem

$$\begin{cases} u_t = L(u) & \text{in} \quad \mathbb{C}^2 \times (0, +\infty) \\ u = g & \text{on} \quad \mathbb{C}^2 \times \{0\} \end{cases}$$

u constant for $|z|^2 + t \gg 0$ [ST_1].

We will complete the proof by proving the following claim: there is t° such that $u(z^\circ, t) < 0$ for $0 < t < t^\circ$.

Suppose not. Then there is a sequence of time instants $t^n \searrow 0$ such that $u(z^\circ, t^n) > 0$. Choose now a neighborhood V of z° such that $\bar{V} \subset B(z^\circ, r)$. Let

$$-m = \max_{z \in bV} (g(z) - v(z)).$$

By (2), $m > 0$. Then there is $\tilde{t} > 0$ such that

(3) $u(z, t) - v(z) \leqslant -m/2$, for $0 \leqslant t \leqslant \tilde{t}$, $z \in bV$

(4) $u(z^\circ, t) - v(z) \geqslant -m/4$, for $0 \leqslant t \leqslant \tilde{t}$.

Let now

$$\chi(t) = \max_{z \in \bar{V}} (u(z, t) - v(z))$$

for $0 \leqslant t \leqslant \bar{t}$. Then χ: $[0, \tilde{t}] \to \mathbb{R}$ is a continuous function with the following properties:

(5) $\chi(0) = 0$, $\chi(t^n) \geqslant 0$, for $n > N$, $u(z, t) \leqslant v(z) + \chi(t)$, $(z, t) \in \bar{V} \times [0, \tilde{t}]$.

For every $0 \leqslant t \leqslant \tilde{t}$, the set

(6) $F_t = \{z \in \bar{V} : u(z, t) = v(z) + \chi(t)\}$ is contained in V.

(The latter condition follows from (3) and (4).)

We use now the following elementary fact: if χ: $[0, \tilde{t}] \to \mathbb{R}$ is a continuous function which is not strictly decreasing, then there is a C^∞ function $q = q(t)$ such that

$$q(t) > \chi(t), \qquad 0 \leqslant t \leqslant \tilde{t}, \qquad q(t^*) = \chi(t^*)$$

for some $t^* \in (0, t)$, $g'(t^*) \geqslant 0$.

Clearly our function χ satisfies the assumption of the assertion.

Let now $\psi = v + q$. By the above inequalities $\psi(z, t) \geqslant u(z, t)$ in $W = V \times (0, \bar{t})$. Let z^* be a point in F_{t*}; by (6), $(z^*, t^*) \in W$ and $\psi(z^*, t^*) = u(z^*, t^*)$. Thus ψ is a C^∞ (viscosity) test function for the solution u of the parabolic equation. Hence we must have $\psi_t(z^*, t^*) \leqslant L(\psi)(z^*, t^*)$. However, $L(\psi)(z^*, t^*) = L(v)(z^*) < 0$ by (1) while $\psi_t(z^*, t^*) = q'(t^*) \geqslant 0$. This is a contradiction, which yields the claim, and hence the theorem. ▮

In this paper we are dealing with the evolution of special subsets, namely Hartogs sets (we recall that a compact set $K \subset \mathbb{C}^2$ is called *Hartogs* if its fibres with respect to the first projection are closed discs centered at 0, if nonempty) and convex sets. The main conclusion of the paper (Corollary 2.9) asserts that the evolution of $b\Omega$, the boundary of a bounded convex domain in $\mathbb{C}^2$ is strictly contracting (i.e., $\Gamma_t = \mathscr{E}_t^{\mathscr{L}}(b\Omega) \subset \Omega$, $t > 0$) and is of stationary type, i.e., $\Gamma_{t_1} \cap \Gamma_{t_2} = \varnothing$, $t_1 \neq t_2$.

We recover here, for the evolution by Levi form, the result which holds for the evolution by mean curvature [H, ES].

The auxiliary tools developed to establish this seem to be of independent interest.

In Section 1 the evolution of a Hartogs set is partially characterized in terms of the parabolic equation

$$v_t = (1 + 4e^{-2v}\,|v_z|^2)^{-1}\, v_{z\bar{z}},$$

where $z = z_1$, $z_2 = w$ and $\exp(-v(z, w))$ stands for the radius of the evolving sets (Lemma 1.1). Comparison with the classical heat equation yields then that the evolution of the bidisc is strictly contracting (Lemma 1.3), in particular flat complex discs disappear instantaneously.

For the sake of completeness we included in Section 1 a result of regularity (for short time) for the parabolic problem

$$\begin{cases} v_t = (1 + 4e^{-2v}\,|v_z|^2)^{-1}\, v_{z\bar{z}} & \text{in} \quad \mathbb{C}^2 \times (0, +\infty) \\ u = g & \text{on} \quad \mathbb{C}^2 \times \{0\}. \end{cases}$$

We are indebted to A. Lunardi for the proof.

In order to apply the results obtained in Section 1 to the evolution of general convex sets we need certain invariance properties of the parabolic Eq. (★) which we study in Section 2.

Strictly speaking, Eq. (★) is invariant only with respect to unitary transformations of the space variables. Realizing, however, that the operator $L(u)$ can be represented as the quotient of the form $\partial\bar{\partial}u \wedge \partial u \wedge \bar{\partial}u$, which is biholomorphically invariant, by $|\partial u|^2\, dz \wedge d\bar{z} \wedge dw \wedge d\bar{w}$ it is possible to show that transformations $(x, t) \mapsto (F(x), ct)$, where F is locally biholomorphic and $c \in \mathbb{R}$, leave invariant the family of subsolutions of $u_t = L(u)$ provided F has a bounded Lipschitz constant λ, $0 < \lambda < 1$, and $|c| \leqslant \lambda^{-6}/2$.

1. PRELIMINARIES ON EVOLUTION OF HARTOGS SETS

We refer to [ST$_3$] for the notion of weak solution of $u_t = L(u)$.

The following lemma simplifies (in some case) drawing the conclusions from the notion of weak solution:

LEMMA 1.1. *Let u be a weak supersolution of $u_t = L(u)$ in the open subset $W \subset \mathbb{C}^2 \times (0, +\infty)$. Let $x = (z, w)$ denote the generic point of $\mathbb{C}^2$ and let ϕ be a C^2 function defined in a neighbourhood $V \subset W$ of $(x_\circ, t_\circ)$ such that $\phi(x_\circ, t_\circ) = u(x_\circ, t_\circ) = A$ and*

$$\{(x, t) \in V : \phi(x, t) > A\} \subseteq \{(x, t) \in V : u(x, t) > A\}. \tag{1.1}$$

Then

$$\phi_t(x_\circ, t_\circ) \geqslant L(\phi)(x_\circ, t_\circ)$$

if $\partial\phi(x_\circ, t_\circ) \neq 0$ or

$$\phi_t(x_\circ, t_\circ) \geqslant (\delta_{\alpha\beta} - \bar{\eta}^\alpha \eta^\beta)\, \phi_{\alpha\bar{\beta}}(x_\circ, t_\circ)$$

for some $\eta \in \mathbb{C}^2$ with $|\eta| \leqslant 1$, if $\partial\phi(x_\circ, t_\circ) = 0$.

Proof. The idea of the proof is to compare u with a nondecreasing continuous function ρ: $\mathbb{R} \to \mathbb{R}$ such that $\rho(A) = A$ and the inequality

$$\phi(x, t) \leqslant (\rho \circ u)(x, t) \tag{1.2}$$

holds in a neighbourhood of $(x_\circ, t_\circ)$.

Since $\rho \circ u$ is again a weak solution the conclusion concerning ϕ is immediate. It remains to construct ρ.

Informally speaking, we have to concern the inclusion (1.1) into the inequality (1.2). Choose a compact neighbourhood N of $(x_\circ, t_\circ)$ such that $N \subset V \subset W$. Let $\rho_1(s) = A$ for $s \leqslant A$. For

$$A \leqslant s \leqslant s_\infty := \sup\{u(x, t) : (z, t) \in N\}$$

let

$$R_s = \{(x, t) \in N : u(x, t) \leqslant s\}.$$

Since u is lower semicontinuous, the R_s's are compact and $R_s \subset R_{s'}$, if $s \leqslant s'$. We let

$$\rho_1(s) = \max\{\phi(x, t) : (x, t) \in R_s\},$$

for $A \leqslant s < s_\infty$. It is clear that ρ_1 is a nondecreasing upper semicontinuous function, by observing that $s \mapsto R_s$ is an upper semicontinuous correspondence.

Observe now that

$$\phi(x, t) \leqslant (\rho_1 \circ u)(x, t),$$

for $(x, t) \in N$.

Suppose for some $(x, t) \in N$, $\phi(x, t) > (\rho_1 \circ u)(x, t)$. If $\phi(x, t) \leqslant A$, this is impossible as $\rho_1 \geqslant A$ always. If $\phi(x, t) > A$, by (1.1) $u(x, t) > A$.

Let $s =: u(x, t)$. Then $(x, t) \in R_s$ and so $\rho_1(s) \geqslant \phi(x, t)$, i.e., $(\rho_1 \circ u)(x, t) \geqslant \phi(x, t)$.

Choose finally a continuous nondecreasing function $\rho: \mathbb{R} \to \mathbb{R}$ such that $\rho \geqslant \rho_1$, $\rho(A) = A$. Then $\phi(x, t) \leqslant (\rho \circ u)(x, t)$. (Note that ρ can be chosen continuous because $\lim_{s \to 0^+} \rho_1(s) = A$.) ▌

Given a Hartogs set $K \subset \mathbb{C}^2$ let $K_z = \{w \in \mathbb{C} : (z, w) \in K\}$ be its generic fibre. By definition K_z is a closed disc centered at 0, if nonempty. Let $e^{-g(z)}$ be its radius, hence

$$K = \{(z, w) \in F \times \mathbb{C} : |w| \leqslant e^{-g(z)}\}, \tag{1.3}$$

where $g: F \to (-\infty, +\infty]$ and $F = pr_1(K)$ (projection on the first coordinate, $pr_1(z, w) = z$) is a compact. By the compactness of K, g is a lower semicontinuous function.

Since the evolution equation $u_t = L(u)$ is invariant with respect to the rotations $(z, w, t) \mapsto (z, e^{i\theta}w, t)$, $\theta \in \mathbb{R}$, it is clear that the evolving sets $K_z^t := \mathscr{E}_t^{\mathscr{L}}(K_z)$ are circled in w; consequently their polynomial hulls $\hat{K}_z^t$ are discs.

(Although it seems probable, we do not know as yet whether the fibres K_z^t themselves are discs.)

We want to obtain an evolution equation for their radii; thus we denote

$$\hat{K}_z^t = \{z\} \times \bar{D}(0, e^{-\chi(z, t)}) \tag{1.4}$$

for $z \in F^t$, where $F^t = pr_1(K^t)$, $K^t = \mathscr{E}_t^{\mathscr{L}}(K)$.

Lemma 1.2. *Let K be a compact Hartogs set in $\mathbb{C}^2$. With the notations* (1.3), (1.4), *let*

$$F^* = \{(z, t) \in \mathbb{C} \times [0, +\infty) : 0 \leqslant t \leqslant T, z \in F^t\},$$

where T denotes the extinction time of K, and

$$\tilde{\chi}(z, t) := \begin{cases} \chi(z, t) & \text{if} \quad (z, t) \in F^* \\ +\infty & \text{if} \quad (z, t) \in \mathbb{C} \times [0, +\infty) \setminus F^*. \end{cases}$$

Then F^* is a compact subset and $\tilde{\chi}$ is a lower semicontinuous function. Furthermore,

$$\tilde{\chi}(z, 0) := \tilde{g}(z) := \begin{cases} g(z) & \text{if } \; z \in F \\ +\infty & \text{if } \; z \in \mathbb{C} \backslash F, \end{cases}$$

and $\tilde{\chi}$ is a weak supersolution of the equation

$$v_t = (1 + 4e^{-2v}\,|v_z|^2)^{-1}\, v_{z\bar{z}}. \tag{1.5}$$

Proof. Recall that $\tilde{\chi}$ is a weak supersolution of (1.5) if for every $(z_\circ, t_\circ)$, $t_\circ > 0$, and a C^2 smooth function $\psi(z, t)$, defined in a neighborhood $V_\circ$ of $(z_\circ, t_\circ)$, and such that

$$\psi(z_\circ, t_\circ) = \tilde{\chi}(z_\circ, t_\circ) \tag{1.6}$$

$$\psi(z, t) \leqslant \tilde{\chi}(z, t), \tag{1.7}$$

for $(z, t) \in V_\circ$, it holds

$$\psi_t \geqslant (1 + 4e^{-2\psi}\,|\psi_z|^2)^{-1}\, \psi_{z\bar{z}} \tag{1.8}$$

at $(z_\circ, t_\circ)$.

Let u: $\mathbb{C}^2 \times [0, +\infty) \to \mathbb{R}$ be a continuous weak solution of $u_t = L(u)$ in $\mathbb{C}^2 \times (0, +\infty)$, with the following properties: $u(x, t) \geqslant 0$ and $u(x, t) = \text{const}$ for $|x| + t$ sufficiently large ($x = (z, w)$).

Let

$$\begin{aligned} K^* &:= \{(x, t) \in \mathbb{C}^2 \times [0, +\infty) : u(x, t) = 0\} \\ &= \{(x, t) \in \mathbb{C}^2 \times [0, +\infty) : 0 \leqslant t \leqslant T, x \in \mathscr{E}_t^{\mathscr{L}}(K)\}. \end{aligned}$$

This set is compact; with $p(z, w, t) = (z, t)$, we have $F^* = p(K^*)$; hence F^* is compact. Furthermore,

$$e^{-\chi(z, t)} = \sup\{|w| : (z, w, t) \in K^*\}.$$

This easily implies that $(z, t) \to e^{-\chi(z, t)}$ is an upper semicontinuous function on F^*; hence $\chi(z, t)$ is lower semicontinuous on F^* and its (trivial) extension by $+\infty$ on $\mathbb{C} \times [0, +\infty) \backslash F^*$ is lower semicontinuous as well (F^* being compact).

To show that $\tilde{\chi}$ is a weak supersolution of (1.5) consider now a function ψ as in (1.6), (1.7). Let further

$$V = \{(z, w, t) \in \mathbb{C}^2 \times \mathbb{R} : (z, t) \in V_\circ, w \in \mathbb{C}^*\}$$

and $\phi(z, w, t) = \psi(z, t) + \log |w|$. Then $\phi \in C^2(V)$, and

$$\phi(z_\circ, w_\circ, t_\circ) = \psi(z_\circ, t_\circ) + \log |w_\circ| = 0,$$

where $w_\circ = e^{-\chi(z_\circ, t_\circ)}$, and

$$\phi(z, w, t) \leqslant \chi(z, t) + \log |w| \tag{1.9}$$

on V.

We will show now that

$$\{(z, w, t) \in V : \phi(z, w, t) > 0\} \subseteq \{(z, w, t) \in V : u(z, w, t) > 0\}. \tag{1.10}$$

Fix $(z, w, t) \in V$ with $\phi(z, w, t) > 0$. We have to show that $u(z, w, t) > 0$ also.

This is trivial when $z \notin F^t$, because then $(z, w) \notin K^t$, hence $u(z, w, t) > 0$. Assume then $z \in F^t$. Because of (1.9), $\chi(z, t) + \log |w| > 0$, i.e., $|w| > e^{-\chi(z, t)}$, and so $w \notin K^t_z$. Hence $u(z, w, t) > 0$.

Having established the inclusion (1.10), we obtain by Lemma 1.1 that

$$\phi_t(z_\circ, w_\circ, t_\circ) \geqslant L(\phi)(z_\circ, w_\circ, t_\circ), \tag{1.11}$$

observing that

$$\partial\phi(z_\circ, w_\circ, t_\circ) = (\psi_z, 1/2\bar{w}_\circ) \neq (0, 0).$$

Substituting to (1.11), $\phi_{z\bar{z}}(z_\circ, w_\circ, t_\circ) = \psi_{z\bar{z}}(z_\circ, t_\circ)$, $\phi_z(z_\circ, w_\circ, t_\circ) = \psi_z(z_\circ, t_\circ)$, $\phi_w(z_\circ, w_\circ, t_\circ) = 1/2w_\circ = e^{\chi(z_\circ, t_\circ)}/2$, we obtain eventually the inequality (1.8). ∎

LEMMA 1.3. *Let* $r > 0$, $R > 0$, *and*

$$K = \bar{D}(0, R) \times \bar{D}(0, r).$$

Then for every $t > 0$,

$$\mathscr{E}^{\mathscr{L}}_t(K) \subset \mathbb{C} \times D(0, r).$$

Proof. K is clearly a convex compact Hartogs set of the form

$$K = \{(z, w) \in \mathbb{C}^2 : z \in \bar{D}(0, R), |w| \leqslant e^{-g(z)}\},$$

where $g \colon \bar{D}(0, R) \to (-\infty, +\infty]$ is the constant function $g(z) = -\log r$.

Let now $K^t = \mathscr{E}^{\mathscr{L}}_t(K)$ and let the function $\chi \colon F^* \to (-\infty, +\infty]$ and the sets F^t, $F^* = \bigcup_{0 \leqslant t \leqslant T} F^{\,t} \times \{0\}$ have the same meaning as in Lemma 1.2. By this lemma, the function $\tilde{\chi} \colon \mathbb{C} \times [0, +\infty) \to (-\infty, +\infty]$, defined by

$\tilde{\chi} = \chi$ on F^* and $\tilde{\chi} = +\infty$ elsewere, is a weak supersolution of the parabolic problem

$$\begin{cases} \chi_t = (1 + 4e^{-2\chi}\, |\chi_z|\, d^2)^{-1}\, \chi_{z\bar{z}} & \text{in} \quad \mathbb{C}^2 \times (0, +\infty) \\ \chi = \tilde{g} & \text{on} \quad \mathbb{C}^2 \times \{0\}, \end{cases} \tag{1.12}$$

where $\tilde{g}(z) = g(z)$ for $z \in \bar{D}(0, R)$ and $\tilde{g}(z) = +\infty$ for $|z| > R$.

We have to show that $\chi(z, t) > -\log r$, for every $t > 0$, $(z, t) \in F^*$. To show this we compare the supersolution $\tilde{\chi}$ with a smooth subsolution of the same equation, which we will construct now by using the standard heat equation.

Assertion. There is a C^∞ smooth subharmonic function $h\colon \mathbb{C} \to \mathbb{R}$ such that $h(z) = -\log r$, for $|z| \leqslant R$, $h(z) = \log|z| + C$, C a constant, for $|z| \geqslant R^*$ where $R^* > R$, $h(z) \geqslant -\log r$ for all $z \in \mathbb{C}$.

The function $H(z, t) = (h * S_t)(z)$, $z \in \mathbb{C}$, $t > 0$, where

$$S_t(z) = (4\pi t)^{-1} \exp(-|z|^2/4t)$$

is continuous in $\mathbb{C} \times [0, +\infty)$ and is a solution of the parabolic problem

$$\begin{cases} H_t = H_{z\bar{z}} & \text{in} \quad \mathbb{C}^2 \times (0, +\infty) \\ H = h & \text{on} \quad \mathbb{C}^2 \times \{0\}. \end{cases} \tag{1.13}$$

Note that the convolution $h * S_t$ is well defined because for every $\varepsilon > 0$ there is a constant $M_\varepsilon > 0$, such that

$$|h(z)| \leqslant M_\varepsilon \exp \varepsilon\, |z|^2, \qquad z \in \mathbb{C};$$

furthermore $H(z, t)$ is (continuous on $\mathbb{C} \times [0, +\infty)$ and) C^∞ for $t > 0$. (See [F] for background and similar results.)

Seeing that $\partial h/\partial z \in L^\infty(\mathbb{C})$, and

$$\frac{i}{2} \int_{\mathbb{C}} dz \wedge d\bar{z} = 1, \qquad S_t > 0,$$

we have $H_z(z, t) = (h_z * S_t)(z)$, and so

$$\sup_{z \in \mathbb{C},\, t \geqslant 0} |H_z(z, t)| \leqslant m < +\infty.$$

Since $H_{z\bar{z}}(z, t) = (h_{z\bar{z}} * S_t)(z)$, and $H_{z\bar{z}} \geqslant 0$, while $S_t(z) > 0$ for all $z \in \mathbb{C}$, $t > 0$, we get $H_{z\bar{z}}(z, t) > 0$ for $(z, t) \in \mathbb{C} \times (0, +\infty)$. Consequently $H_t(z, t)$ is positive for $t>0$ and so $H(z, t)$ is strictly increasing in t for $(z, t) \in \mathbb{C} \times (0, +\infty)$. Thus

$$H(z, t) > \log r \qquad \text{for} \quad (z, t) \in \mathbb{C} \times (0, +\infty). \tag{1.14}$$

Consider the family of functions

$$H^{\alpha, s}(z, t) := H(z, st) - \alpha, \qquad (z, t) \in \mathbb{C} \times [0, +\infty)$$

with $\alpha \geqslant 0$, $s>0$. We will identify now a sufficient condition for α, s under which $H^{\alpha, s}$ is a subsolution of (1.12). Since

$$\begin{aligned} \frac{\partial H^{\alpha, s}}{\partial t}(z, st) &= s\frac{\partial H}{\partial t}(z, st) = sH_{z\bar{z}}(z, st) \\ &= k(1+4e^{-2H^{\alpha, s}(z, t)}\,|H_z^{\alpha, s}(z, t)|^2)^{-1}\,H_{z\bar{z}}^{\alpha, s}(z, t), \end{aligned}$$

where

$$k = s(1+4e^{-2H(z, st)}e^{2\alpha}\,|H_z(z, t)|^2),$$

we obtain that $H^{\alpha, s}$ is a (classical) subsolution of (1.12), provided $k \leqslant 1$. (Note that $H_{z\bar{z}} \geqslant 0$.) Clearly $k \leqslant s(1+4r^{-2}e^{2\alpha}m^2)$. Thus if we let $s = s(\alpha) = (1+4r^{-2}e^{2\alpha}m^2)^{-1}/2$, for $\alpha \geqslant 0$, and

$$H^{\alpha}(z, t) := H^{\alpha, s(\alpha)}(z, t) = H(z, s(\alpha)\, t) - \alpha,$$

for $(z, t) \in \mathbb{C} \times [0, +\infty)$, we obtain that the H^{α}'s, $\alpha \geqslant 0$, are strict subsolutions of (1.12), i.e.,

$$H_t^{\alpha}(z, t) < (1+4e^{-2H^{\alpha}(z, t)}\,|H_z^{\alpha, s}(z, t)|^2)^{-1}\,H_{z\bar{z}}^{\alpha}(z, t),$$

for $(z, t) \in \mathbb{C} \times (0, +\infty)$, because $k \leqslant 1/2$ and $H_{z\bar{z}}(z, t) > 0$, for $t>0$. This family is continuous in α, with respect to the topology of uniform convergence on compact subsets of $\mathbb{C} \times [0, +\infty)$. It is clear that if $\alpha_\circ > 0$ is large enough, then

$$\tilde{\chi}(z, t) > H^{\alpha_\circ}(z, t) \tag{1.15}$$

for $(z, t) \in \mathbb{C} \times [0, +\infty)$. We claim that $H^{\alpha_\circ} < \tilde{\chi}$ on $\mathbb{C} \times [0, +\infty)$ for all $\alpha_\circ \geqslant 0$.

Suppose this is false and take α to be the infimum of all values of $\alpha_\circ > 0$ for which (1.16) is true. Then we have

$$H^{\alpha}(z, t) \leqslant \tilde{\chi}(z, t) \tag{1.16}$$

for $(z, t) \in \mathbb{C} \times [0, +\infty)$ and

$$H^\alpha(z_\circ, t_\circ) = \tilde{\chi}(z_\circ, t_\circ) \tag{1.17}$$

for some $(z_\circ, t_\circ) \in \mathbb{C} \times [0, +\infty)$.

Suppose $t_\circ > 0$. Since $\tilde{\chi}$ is a weak supersolution of (1.12), by Lemma 1.2, and H^α is a C^∞ test function on $\mathbb{C} \times (0, +\infty)$, we get

$$H_t^\alpha(z_\circ, t_\circ) \geqslant (1 + 4e^{-2H^\alpha(z_\circ, t_\circ)} |H_z^\alpha(z_\circ, t_\circ)|^2)^{-1/2} H_{z\bar{z}}^\alpha(z_\circ, t_\circ)$$

which contradicts (1.15).

Thus $t_\circ = 0$. Suppose $\alpha > 0$. Then $H^\alpha(z_\circ, 0) \leqslant \log r - \alpha < \tilde{\chi}(z_\circ, 0)$, which is a contradiction. Thus $\alpha = 0$ and, by (1.17),

$$H^0(z, t) \leqslant \tilde{\chi}(z, t)$$

for $(z, t) \in \mathbb{C} \times [0, +\infty)$. But by (1.14), $H^0(z, t) > \log r$ for $t > 0$, i.e., $\tilde{\chi}(z, t) > \log r$ for $t > 0$. ∎

Remark 1.1. As we will see later on, more is true, namely $\mathscr{E}_t^{\mathscr{L}}(K) \subset \mathring{K}$ for $t > 0$.

As a corollary we obtain that flat complex discs disappear instantaneously:

COROLLARY 1.4. *Let Δ be a closed disc in a complex line in $\mathbb{C}^2$. Then its extinction time is* 0.

Proof. Let R be the radius of Δ. By combination of translation and unitary transformation (which commute with evolution by Levi curvature) we bring Δ to be $\bar{D}(0, R) \times \{r\} \subseteq bK$ where K is as in Lemma 1.3. Since Δ is polynomially convex, $\mathscr{E}_t^{\mathscr{L}}(\Delta) \subset \Delta \subset bK$. By Lemma 1.3

$$\mathscr{E}_t^{\mathscr{L}}(\Delta) \subset \mathscr{E}_t^{\mathscr{L}}(K) \subset \mathbb{C} \times D(0, r)$$

for $t > 0$, i.e., $\mathscr{E}_t^{\mathscr{L}}(\Delta) \cap \Delta = \varnothing$ for $t > 0$.

Thus $\mathscr{E}_t^{\mathscr{L}}(\Delta) = \varnothing$ for $t > 0$. ∎

We end this section by proving a regularity theorem for the equation

$$u_t = (1 + 4e^{-2u} |u_z|^2)^{-1} u_{z\bar{z}}.$$

More generally we consider the parabolic problem

$$\begin{cases} u_t = f(u, Du)\, \Delta u & \text{for} \quad t \geqslant 0, x \in \mathbb{R}^n \\ u = u_\circ & \text{for} \quad x \in \mathbb{R}^n, \end{cases} \tag{P}$$

where $Du = (D_1 u, ..., D_n u)$, $D_j = \partial/\partial x_j$ and Δ is the Laplace operator $\sum_{j=1}^n D_j^2$.

Let us denote by $C_b^\infty(\mathbb{R}^n)$ the space of smooth functions on $\mathbb{R}^n$ bounded with all derivatives.

THEOREM 1.5. *Assume that* $f: \mathbb{R}^{n+1} \to \mathbb{R}$ *is* C^∞, $f > 0$ *and* $u_\circ \in C_b^\infty(\mathbb{R}^n)$. *Then there exists* $\tau > 0$ *and a unique solution* $u \in C^{1,2}([0, \tau) \times \mathbb{R}^n)$ *of* (P) *such that* u, u_t, $D_i u$, $D_{ij} u$ *are bounded in* $[0, T] \times \mathbb{R}^n$ *for all* $T < \tau$. *Moreover* $u \in C_b^\infty([0, T] \times \mathbb{R}^n)$ *for all* $T < \tau$.

Proof. We embed the problem in $X = BUC(\mathbb{R}^n)$, the space of uniformly continuous, bounded functions $\mathbb{R} \to \mathbb{R}$. Namely letting $u(t)$ denote the function $t \mapsto u(t, \cdot)$ we consider the problem

$$\begin{cases} u'(t) = \varphi(u(t))\, Au(t) & \text{for} \quad t \geqslant 0 \\ u(0) = u_\circ, \end{cases} \tag{P$'$}$$

where $A: \mathscr{D}(A) \to X$ is the realization of Δ in X and $\varphi(x) = f(v(x), Dv(x))$ for v belonging to $BUC^1(\mathbb{R}^n) := \{v \in BUC(\mathbb{R}^n) : Dv \in BUC(\mathbb{R}^n)\}$. φ is smooth from $BUC^1(\mathbb{R}^n)$ to X; consequently $v \mapsto \varphi(v) \cdot A$ is smooth from $BUC^1(\mathbb{R}^n)$ to $L(\mathscr{D}(A), X)$, the space of all linear operators $\mathscr{D}(A) \to X$. A generates an analytic semigroup and therefore is a sectorial operator in view of Stewart Theorem [St; L, Corollary 3.1.9]. Moreover $\mathscr{D}(-A^\alpha)$ and every interpolation space $\mathscr{D}_A(\alpha, p)$, $p \geqslant 1$ are continuously embedded in $BUC^1(\mathbb{R}^n)$ provided $\alpha > 1$ [L, Theorem 3.1.12 and Proposition 2.2.15] and, finally, $u_\circ \in \mathscr{D}(A)$ (and even $u_\circ \in \mathscr{D}(A^k)$ for every $k \in \mathbb{N}$).

In these conditions we invoke the existence theorem of Sobolevskiĭ [S]: there exists a maximal $\tau > 0$ and a unique $u \in C^1([0, \tau); X) \cap C^0([0, \tau); \mathscr{D}(A))$ which solves (P') in $[0, \tau)$.

We observe that, since $u \in C^0([0, \tau); \mathscr{D}(A))$, the function $(t, x) \mapsto \Delta u(t, x)$, $u(t, x) := u(t)(x)$, is continuous in $[0, \tau) \in \mathbb{R}^n$ and $(t, x) \mapsto u(t, x)$ is a solution of (P').

We are going to prove the regularity of u by induction, in view of the following theorem [LSU, Theorem 5.1]: let l be a positive integer, $a \in C^{l/2, l}([0, T] \times \mathbb{R}^n)$, $a = a(t, x) > 0$ and $u_\circ \in C^{l+2}(\mathbb{R}^n)$; then the solution of the parabolic problem

$$\begin{cases} u_t = a(t, x)\, \Delta u(t, x) & \text{for} \quad t \in [0, T] \ x \in \mathbb{R}^n \\ u(o, x) = u_\circ(x) & \text{for} \quad x \in \mathbb{R}^n \end{cases}$$

belongs to $C^{1 + l/2, 2 + l}([0, T] \times \mathbb{R}^n)$ with norm less than $c(l)\, \|u_\circ\|_{C^{2+l}}$. Let $k = 1$. In our case, since $t \mapsto u(t) \in C^1([0, \tau); X) \cap C^0([0, \tau); \mathscr{D}(A))$ we have $u \in C^{1/2}([0, T]; BUC^1(\mathbb{R}^n))$. This follows from [L, Proposition 1.1.4, (i)]. Indeed by [L, Proposition 3.1.11]

$$\|\varphi\|_{C^1} \leqslant c\, \|\varphi\|_\infty^{1/2}\, \|\varphi\|_{\mathscr{D}(A)}^{1/2}$$

for every $\varphi \in \mathscr{D}(A)$. Moreover, since $\mathscr{D}(A)$ is continuously embedded in every space $C^{1+\alpha}(\mathbb{R}^n)$ for all $\alpha \in (01)$, $D_j u \in C^{\alpha/2, \alpha}([0, T] \times \mathbb{R}^n)$ and consequently $(t, x) \mapsto f(u(t, x), Du(t, x))$ belongs to $C^{\alpha/2, \alpha}([0, T] \times \mathbb{R}^n)$. Thus, since $u_\circ \in C^{2+\alpha}(\mathbb{R}^n)$ the quoted theorem of [LSU] applies and yields $u \in C^{1+\alpha/2, 2+\alpha}([0, T] \times \mathbb{R}^n)$.

Now let us assume that the statement is true for a $k \geqslant 1$ and prove it for $k+1$.

Since $u \in C^{k+\alpha/2, 2k+\alpha}([0, T] \times \mathbb{R}^n)$, $D_j u \in C^{k-1/2+\alpha/2, 2k-1+\alpha}([0, T] \times \mathbb{R}^n)$, $j = 1, ..., n$ [LSU]; consequently $f(u, Du) \in C^{k-1/2+\alpha/2, 2k-1+\alpha}([0, T] \times \mathbb{R}^n)$. Again, by the quoted theorem of [LSU] applied with $l = 2k + \alpha - 1$ we have $u \in C^{k+1+\alpha/2, 2k+2+\alpha}([0, T] \times \mathbb{R}^n)$ as desired.

Finally to show unicity we consider a solution $\tilde{u} \in C^{1,2}([0, \tau) \times \mathbb{R}^n)$ of (P) such that $\tilde{u}, \tilde{u}_t, D_i \tilde{u}, D_{ij} \tilde{u}$ are bounded in $[0, T] \times \mathbb{R}^n$ for every $T < \tau$. We have to prove that $\tilde{u}$ coincides with u. This is actually true since we can argue for $\tilde{u}$ in the same way as for u to conclude that $\tilde{u} \in C^\infty([0, T] \times \mathbb{R}^n)$ and all its derivatives are bounded for every $T < \tau$. In particular, $t \mapsto \tilde{u}(t, \cdot)$ belongs to $C^1([0, T]; X) \cap C^0([0, T]; \mathscr{D}(A))$ for every $T < \tau$ and therefore coincides with u since in this class we have unicity. ∎

2. SEMI-INVARIANCE OF THE EVOLUTION EQUATION

We study now the invariance with respect to locally biholomorphic maps.

PROPOSITION 2.1. *Let U, W be open subsets of $\mathbb{C}^2$ and v a weak subsolution of $v_t = L(v)$ on $W \times (0, +\infty)$, with values in $[-\infty, +\infty)$. Let $F\colon U \to W$ be a locally biholomorphic map. Assume that F is locally biLipschitz with constant λ, i.e., locally*

$$\lambda^{-1} |\xi| \leqslant |\partial F(\xi)| \leqslant \lambda |\xi|$$

for $\xi \in \mathbb{C}^2$. Assume further that v is nondecreasing in time, i.e., for all $x = (z, w) \in U$, $t_1 \leqslant t_2$ in $\mathbb{R}$, $v(x, t_1) \leqslant v(x, t_2)$. Then the function $u(x, t) = v(F(x), ct)$ is a weak subsolution of $v_t = L(v)$ provided $c \leqslant \lambda^{-6}/2$.

Proof. We have to check that u satisfies the definition of weak subsolution. Let ϕ be a C^2 function in a neighbourhood of $(x_\circ, t_\circ) \in U \times (0, +\infty)$ satisfying $\phi(x_\circ, t_\circ) = u(x_\circ, t_\circ)$, $\phi(x, t) \leqslant u(x, t)$. Let $y_\circ = F(x_\circ)$, $s_\circ = ct_\circ$. Let G be a local inverse to F near $y_\circ = F(x_\circ)$. We let $\psi(y, s) = \phi(G(y), c^{-1}s)$. Then ψ is a C^2 function near $(y_\circ, s_\circ)$ such that $\psi(y_\circ, s_\circ) = v(y_\circ, s_\circ)$, $\psi(y, s) \leqslant v(y, s)$.

Consider first the case when $\partial\phi(x_\circ, t_\circ) \neq 0$. Since v is a weak subsolution we obtain

$$\psi_s(y_\circ, s_\circ) \leqslant L(\psi)(y_\circ, s_\circ). \tag{2.1}$$

Note that since v is nondecreasing in the time variable s, we have $\psi_s(y_\circ, s_\circ) \geqslant 0$, and consequently $L(\psi)(y_\circ, s_\circ) \geqslant 0$. Thus in the following inequalities the direction is preserved when multiplication by a nonnegative factor is involved. Now $\phi_t(x_\circ, t_\circ) = c\psi_s(y_\circ, s_\circ)$. Denote by ω the form $dz \wedge d\bar{z} \wedge dw \wedge d\bar{w}$ and by η the form $d\xi \wedge d\bar{\xi} \wedge d\zeta \wedge d\bar{\zeta}$ where $y = (\xi, \zeta)$. Clearly $\eta = |J(F)|^2\, \omega$ where $J(F)$ denotes the Jacobian determinant of F.

The following calculations relate the quantities $L(\psi)$ and $L(\phi)$,

$$\begin{aligned}
L(\psi)\,\eta &= |\partial\psi|^{-2}\, \partial\bar{\partial}\psi \wedge \partial\psi \wedge \bar{\partial}\psi \\
&= |\partial(\phi \circ G)|^{-2}\, G^*(\partial\bar{\partial}\phi \wedge \partial\phi \wedge \bar{\partial}\phi) \\
&= |\partial(\phi \circ G)|^{-2}\, |\partial\phi|^2\, L(\phi \circ G)\, G^*(\omega) \\
&= |\partial(\phi \circ G)|^{-2}\, |\partial\phi|^2\, L(\phi \circ G)\, |J(G)|^2 \eta.
\end{aligned}$$

Thus we obtain, with $G(y_\circ) = x_\circ$,

$$L(\psi)(y_\circ, s_\circ) = L(\phi)(x_\circ, t_\circ)\, Q, \tag{2.2}$$

where

$$Q := |J(G)(y_\circ)|^2\, |\partial(\psi \circ F)(x_\circ, t_\circ)|^2\, |\partial\psi(F(x_\circ), t_\circ)|^{-2} \leqslant \lambda^6.$$

Then, owing to (2.1) and (2.2), we obtain

$$\begin{aligned}
\phi_t(x_\circ, t_\circ) = c\psi_s(y_\circ, s_\circ) &\leqslant cL(\psi)(y_\circ, s_\circ) \\
&\leqslant (cQ)\, L(\phi)(x_\circ, t_\circ) \leqslant c\lambda^6 L(\phi)(x_\circ, t_\circ) \\
&\leqslant L(\phi)(x_\circ, t_\circ)
\end{aligned}$$

provided $c \leqslant \lambda^{-6}$.

The case when $\partial\phi(x_\circ, t_\circ) = 0$ is similar. Indeed, since $\partial\psi(y_\circ, s_\circ) = 0$, we have

$$\psi(y_\circ, s_\circ) \leqslant (\delta_{\alpha\beta} - \bar{\mu}^\alpha \mu^\beta)\, \psi_{\alpha\bar{\beta}}(y_\circ, s_\circ), \tag{2.3}$$

with some $\mu \in \mathbb{C}^2$, $|\mu| \leqslant 1$.

Let

$$Hess^{\mathbb{C}}\phi_{(x_\circ, s_\circ)} = (\phi_{\alpha\bar{\beta}}(x_\circ, t_\circ)), \qquad Hess^{\mathbb{C}}\psi_{(y_\circ, s_\circ)} = (\psi_{\alpha\bar{\beta}}(y_\circ, s_\circ)),$$

$(\partial G)_{y_\circ}$ be the $\mathbb{C}$-differential of G at $y_\circ$ and $\langle\ ,\ \rangle$ the standard hermitian product in $\mathbb{C}^2$. As

$$Hess^{\mathbb{C}}\psi_{(y_\circ, s_\circ)} = (\partial G)^*_{y_\circ}\, Hess^{\mathbb{C}}\phi_{(x_\circ, t_\circ)}(\partial G)_{y_\circ},$$

and since

$$(\delta_{\alpha\beta} - \bar{\mu}^\alpha \mu^\beta)\, \psi_{\alpha\bar{\beta}}(y_\circ, s_\circ) = \langle Hess^{\mathbb{C}}\psi_{(y_\circ, s_\circ)} p, p\rangle + \langle Hess^{\mathbb{C}}\psi_{(y_\circ, s_\circ)} q, q\rangle,$$

where $p, q \in \mathbb{C}^2$ are such that $I - \mu \otimes \mu = p \otimes p + q \otimes q$, with $p \perp q$, i.e., $p \perp \mu$, $|p| = 1$, $q \in \mathbb{C}\mu$, $|q|^2 = 1 - |\mu|^2$, (2.3) implies

$$\begin{aligned} \phi_t(x_\circ, t_\circ) &\leqslant (\delta_{\alpha\beta} - \bar{\mu}^\alpha \mu^\beta)\, \psi_{\alpha\bar{\beta}}(w_\circ, s_\circ) \\ &\leqslant c\, \langle Hess^{\mathbb{C}}\phi_{(x_\circ, t_\circ)}(\partial G)_{y_\circ} p, (\partial G)_{y_\circ} p\rangle \\ &\quad + c\, \langle Hess^{\mathbb{C}}\phi_{(x_\circ, t_\circ)}(\partial G)_{y_\circ} q, (\partial G)_{y_\circ} q\rangle. \end{aligned} \tag{2.4}$$

Consider the maximum of the last two terms; then the vector $(\partial G)_{y_\circ} p$ or $(\partial G)_{y_\circ} q$ corresponding to it is of the form $a\gamma$, $\gamma \in \mathbb{C}^2$, $a \in \mathbb{C}$, with $|\gamma| = 1$, $|a| \leqslant \lambda$, and

$$\begin{aligned} \phi_t(x_\circ, t_\circ) &\leqslant 2c\lambda^2 \langle Hess^{\mathbb{C}}\phi_{(x_\circ, t_\circ)}\gamma, \gamma\rangle \\ &= 2c\lambda^2 (\delta_{\alpha\beta} - \bar{\chi}^\alpha \chi^\beta)\, \phi_{\alpha\bar{\beta}}(x_\circ, t_\circ) \end{aligned}$$

if $\chi \in \mathbb{C}^2$ is a unit vector complementary to γ. Since $\lambda \geqslant 1$, $c \leqslant \lambda^{-6}/2$ suffices for the inequality

$$\phi_t(x_\circ, t_\circ) \leqslant (\delta_{\alpha\beta} - \bar{\chi}^\alpha \chi^\beta)\, \phi_{\alpha\bar{\beta}}(x_\circ, t_\circ)$$

to hold. ∎

COROLLARY 2.2. *Let $F\colon \Omega \to \Omega^*$, where $\Omega, \Omega^* \subset \mathbb{C}^2$ are open and bounded, be a proper map, locally biholomorphic and locally biLipschitz. Assume that the stationary problem $L(v^*) = 1$ in Ω^*, $v^* = 0$ on $b\Omega^*$ has a weak solution $v^* \in C^0(\bar{\Omega}^*)$. Then the corresponding stationary problem in Ω has a weak solution $u \in C^0(\bar{\Omega})$.*

Proof. As we know [ST_3], v^* is a weak subsolution of the stationary problem if and only if the function $V(y, s) = v^*(y) + s$, $(y, s) \in \Omega^* \times (0, +\infty)$ is a weak subsolution of the parabolic problem. By Proposition 2.1 the function $U(x, t) = V(F(x), t) = v^*(F(x), t) + ct = v(x) + ct$ is, for a small positive c, a weak subsolution of the parabolic equation, seeing that $V(y, s)$ is increasing in time. But if $v(x) + ct$ is a weak subsolution of the parabolic equation, then $(1/c)\, v(x)+t$ is also, and in view of the above-mentioned inequality, $(1/c)\, v$ is a weak subsolution of $L(u)=1$. Observe that, since $F\colon \Omega \to \Omega^*$ is proper and $v^*\colon \bar{\Omega}^* \to (-\infty, 0]$ is continuous and equal 0 on the boundary $b\Omega^*$, $v=v^* \circ F$ is continuous on $\bar{\Omega}$ and equal 0 on its boundary. Thus the functions $(1/c)\, v$ and 0 are, respectively, lower and upper continuous (on $\bar{\Omega}$) barriers for the equation $L(u)=1$ with the same boundary values 0. In this situation the Perron method and Walsh lemma [W] yield a weak solution u of $L(u)=1$ which is continuous on $\bar{\Omega}$ and 0 on $b\Omega$. ∎

LEMMA 2.3. *Let K be a compact subset of $\mathbb{C}^2$. Then K is a Stein compact (i.e., has a Stein neighbourhoods basis) if and only if there is a continuous function $g\colon \mathbb{C}^2 \to \mathbb{R}$ such that*

(i) *$L(g) \geqslant 0$ in the weak sense,*
(ii) *$g^{-1}(0)=K$,*
(iii) *$g \geqslant 0$ on $\mathbb{C}^2$ and $g(x)$ is constant for $|x| \gg 0$.*

Proof. It is clear that the existence of g implies that K is a Stein compact. Conversely, if K is a Stein compact, we can assume, without loss of generality, that there exists a sequence of domains $\{\Omega_n\}_{n \geqslant 1}$ such that

$$\bigcap_{n \geqslant 1} \Omega_n = K, \qquad \bar{\Omega}_{n+1} \subset \Omega_n,$$

for $n \geqslant 1$ and $b\Omega_n$ is C^2 regular and strictly pseudoconvex. To construct g select, for each n, an open neighbourhood V_n of $b\Omega_n$ so that $V_n \cap V_{n+1} = \varnothing$, $\bar{V}_n \subset \Omega_{n+1}$ and $V_n \cap \bar{\Omega}_{n-1} = \varnothing$. Consequently $V_n \cap V_m = \varnothing$ for $n \neq m$. By shrinking, if necessary, V_n's we can obtain that for every n there is a C^2 smooth defining function $\rho_n\colon V_n \to \mathbb{R}$, such that

$$V_n \cap \Omega_n = \{x \in V_n : \rho_n(x) < 0\},$$

ρ_n is strictly p.s.h. and $\partial \rho_n \neq 0$ on $b\Omega_n$.

Choose now a sequence of positive numbers δ_n such that $\sum_{n \geqslant 1} \delta_n < +\infty$ and

$$F_n := \Omega_n \cup \{x \in V_n : \rho_n(x) \leqslant \delta_n\} \subset \Omega_{n-1};$$

in particular F_n is compact.

The function

$$\rho_n^*(x) = \begin{cases} 0 & \text{if } \ x \in \Omega_n \\ \max(0, \rho_n(x)) & \text{if } \ x \in \mathring{F}_n \\ \min(\rho_n(x), \delta_n) & \text{if } \ x \in V_n \backslash \Omega_n \end{cases}$$

is well defined and is clearly a weak subsolution of $L(\rho)=0$ (being so locally). We can define now the function g.

We require that

$$g(x)=\begin{cases}\sum\limits_{n\geqslant 1}\delta_n & \text{if}\quad x\in\mathbb{C}^2\backslash K\\ \sum\limits_{k\geqslant n}\delta_k & \text{if}\quad x\in\Omega_{n-1}\backslash F_n,\quad n\geqslant 2\\ \sum\limits_{k\geqslant n+1}\delta_k+\rho_n^*(x) & \text{if}\quad x\in V_n\\ 0 & \text{if}\quad x\in K.\end{cases}$$

By the whole geometric configuration it is clear that $g\colon\mathbb{C}^2\to\mathbb{R}$ is a continuous function, nonnegative, constant $(=\sum_{n\geqslant 1}\delta_n)$ for $|x|\gg 0$ and that $\{x\in\mathbb{C}^2 : g(x)=0\}=K$, and, first of all, that g is consistently defined. It is also clear that g is a weak subsolution of $L(g)=0$ on $\mathbb{C}^2\backslash K$, because it is defined as a weak subsolution on an open subset of $\mathbb{C}^2\backslash K$. To conclude that g is a weak subsolution on $\mathbb{C}^2$, let $g_n=\max(g,\sum_{n\geqslant k}\delta_k)$. Then g_n is locally a weak subsolution everywhere and $g_n\searrow g$, hence g is a weak subsolution of $L(g)=0$ in $\mathbb{C}^2$. ▌

In order to apply the semi-invariance with respect to biholomorphic maps we must have weak subsolutions of the parabolic problem which are non decreasing in time. The following proposition gives a sufficient condition for such a situation.

Proposition 2.4. *Let $g\colon\mathbb{C}^2\to[0,+\infty)$ be continuous and such that*

$$L(g)\geqslant 0,\qquad g\geqslant 0,\qquad \textit{on } \mathbb{C}^2,\qquad g \textit{ constant for } |x|\gg 0. \tag{2.5}$$

Let $u=u(x,t)$, $(x,t)\in\mathbb{C}^2\times[0,+\infty)$ be a continuous weak solution of the parabolic problem

$$\begin{cases}u_t=L(u) & \textit{in}\quad \mathbb{C}^2\times(0,+\infty)\\ u=g & \textit{on}\quad \mathbb{C}^2\times\{0\}.\end{cases} \tag{2.6}$$

Then for every x the function $t\to u(x,t)$ is nondecreasing. Consequently, for every $t\geqslant 0$, $L(u(\cdot,t))\geqslant 0$, in $\mathbb{C}^2$, in the weak sense.

Corollary 2.5. *Let $K\subset\mathbb{C}^2$ be a compact with a defining function g as in Lemma* 2.3. *Then the evolution of K is weakly decreasing, i.e.,*

$$\mathscr{E}_{t_2}^{\mathscr{L}}(K)\subseteq\mathscr{E}_{t_1}^{\mathscr{L}}(K)\subseteq K$$

if $0 \leqslant t_1 \leqslant t_2$. *Furthermore, each of the compacts* $\mathscr{E}_t^{\mathscr{L}}(K)$, $0 \leqslant t \leqslant T$, *has a defining function with the same properties* (i), (ii), (iii). *In particular each* $\mathscr{E}_t^{\mathscr{L}}(K)$ *is a Stein compact.*

We first derive the corollary from Proposition 2.4.

Proof of the Corollary. Let u be the solution of the parabolic problem (2.6) with the initial data g. Owing to Proposition 2.4, $0 \leqslant g(x) \leqslant u(x, t_1) \leqslant u(x, t_2)$ if $0 \leqslant t_1 \leqslant t_2$, hence

$$\begin{aligned}\mathscr{E}_{t_2}^{\mathscr{L}}(K) &= \{x \in \mathbb{C}^2 : u(x, t_2) \leqslant 0\} \subseteq \{x \in \mathbb{C}^2 : u(x, t_1) \leqslant 0\} \\ &= \mathscr{E}_{t_1}^{\mathscr{L}}(K).\end{aligned}$$

Furthermore

$$\begin{aligned}\mathscr{E}_t^{\mathscr{L}}(K) &= \{x \in \mathbb{C}^2 : u(x, t) \leqslant 0\} \\ &= \bigcap_{\varepsilon > 0} \{x \in \mathbb{C}^2 : u(x, t) \leqslant \varepsilon\}.\end{aligned}$$

Again by Proposition 2.4, $x \mapsto u(x, t)$ is a weak subsolution of $L(v) = 0$, hence the sets $\{x \in \mathbb{C}^2 : u(x, t) \leqslant \varepsilon\}$ are Stein. ∎

Proof of Proposition 2.4. In this proof it will be convenient to consider, weak subsolutions of the equation

$$u_t = L(u) \tag{2.7}$$

in the whole of $\mathbb{C}^2 \times (-\infty, +\infty)$. Observe that, trivially, the function $v = g$ is a continuous weak subsolution of (2.7). Define now, on $\mathbb{C}^2 \times (-\infty, +\infty)$

$$U(x, t) = \begin{cases} v(x, t) & \text{if} \quad t < 0 \\ \max(v(x, t), u(x, t)) & \text{if} \quad t \geqslant 0. \end{cases}$$

Observe first that U is continuous everywhere, also for $t = 0$, because $u(x, 0) = g(x) = v(x, 0)$.

Assertion. U is a weak subsolution of (2.7) on $\mathbb{C}^2 \times (-\infty, +\infty)$. To check this, consider ϕ, C^2 smooth in a neighbourhood of $(x_\circ, t_\circ)$ such that $\phi(x_\circ, t_\circ) = U(x_\circ, t_\circ)$ and $\phi(x, t) \geqslant U(x, t)$.

If $t_\circ \leqslant 0$, this means that $\phi(x_\circ, t) \geqslant g(x_\circ)$, hence $\phi_t(x_\circ, t_\circ) = 0$, and since $\phi(x, t_\circ) \geqslant g(x)$, in a neighbourhood, and g is a weak supersolution of $L(g) = 0$, $L(\phi(x, t_\circ)) \geqslant 0$.

On the other hand, on $\mathbb{C}^2 \times (0, +\infty)$ the function U is the maximum of two weak subsolutions v and u, consequently it is a weak subsolution there. Let C be the positive constant such that $g(x) = C$ for $|x| \gg 0$. Denote

$\Omega = \{x \in \mathbb{C}^2 : g(x) < C\}$. Since $L(g) \geqslant 0$, Ω is pseudoconvex. Since $C = \sup g$, it follows from [ST_3, Remark 2.2] that $u(x, t) = C$ on $(\mathbb{C}^2 \backslash \Omega) \times [0, +\infty)$. On the other hand there is $T_\circ > 0$ such that $u(x, t) = C$ for $t \geqslant T_\circ$, $x \in \mathbb{C}^2$. Combining this we obtain that $U(x, t) = C$, for $x \notin \Omega$ or $t \geqslant T_\circ$.

For $h \geqslant 0$, denote U^h the translation to the right of U, i.e.,

$$U^h(x, t) = U(x, t-h).$$

Clearly U^h is a weak subsolution of (2.7) on $\mathbb{C}^2 \times (-\infty, +\infty)$. Let

$$U^* = \sup_{0 \leqslant h < +\infty} U^h. \tag{2.8}$$

Observe that due to (2.8)

$$U^*(x, t) = \sup\{U^h(x, t) : 0 \leqslant h \leqslant T_\circ\}.$$

Since $\{U^h\}_{0 \leqslant h \leqslant T_\circ}$ is a uniformly continuous (on each compact of $\mathbb{C}^2 \times \mathbb{R}$) family of continuous weak subsolutions of (2.7), therefore U^* is a continuous weak subsolution of (2.7).

The crucial property of the function U^* is that it is nondecreasing in t. It is clear if we rewrite (2.8) as

$$U^*(x, t) = \sup\{U(x, t) : s \leqslant t\}.$$

Denote now $W = \Omega \times (0, T_\circ)$. Then $U^* = g$ on $\Omega \times \{0\}$ while $U^* = C$ on $b\Omega \backslash \mathbb{C}^2 \times \{0\}$, by (2.8). This means that $U^* = u$ on bW. U^* being a weak subsolution, we obtain in view of the comparison principle that $U^* \leqslant u$ in W, and $U^* = u$ outside of W. On the other hand $u \leqslant U \leqslant U^*$ by construction. Hence $u = U^*$ which means that for every x, $t \mapsto u(x, t)$ is nondecreasing. ∎

LEMMA 2.6. *Let $K', K \subset \mathbb{C}^2$ be Stein compacta with Stein neighbourhoods V and W respectively and S a compact subset of bK'. Let F: $V \to W$ be a proper locally biholomorphic map such that $F(K') \subseteq K$ and $F(S) \subseteq bK$. Assume that the evolution of K is strictly contracting (i.e., $\mathcal{E}_t^{\mathcal{L}}(K) \subset \mathring{K}$ for $t > 0$). Then*

$$\mathcal{E}_t^{\mathcal{L}}(K') \cap S = \varnothing$$

for $t > 0$.

Proof. Let g be a defining function for the set K, satisfying the conditions of Lemma 2.3.

By shrinking, if necessary, the sets V, W we can assume that F is locally biLipschitz, with constant $\lambda < +\infty$. Choose $m > 0$ such that $\{g \leqslant m\} \subseteq W$.

The function $g^* = \min(g, m)$ is a weak subsolution of $L(g^*) = 0$. Let $v \in C^0(\mathbb{C}^2 \times [0, +\infty))$ be a weak solution of the parabolic problem

$$\begin{cases} v_t = L(v) & \text{in} \quad \mathbb{C}^2 \times (0, +\infty) \\ u = g^* & \text{on} \quad \mathbb{C}^2 \times \{0\}. \end{cases} \tag{2.9}$$

By Proposition 2.4, v is nondecreasing in s ($L(g^*) \geqslant 0$), and so Proposition 2.1 applies.

Let $u(x, t) = v(F(x), ct)$, $x \in V$, $t \geqslant 0$. We fix $c > 0$ small enough so that $u = u(x, t)$ is a weak subsolution of the equation $v_s = L(v)$ in $V \times (0, +\infty)$, such that $v(x, 0) = g^*(F(x))$, $x \in V$. Clearly $0 \leqslant v(x, 0) \leqslant m$, $x \in V$. To facilitate the comparison with the solutions of the parabolic equation in V we will produce an initial function $\tilde{g}$ constant near bV. Let g' be again a defining function for K', satisfying the conditions of Lemma 2.3 and in addition $g'^{-1}(0) = K'$, $g' \geqslant 0$, $g' = m$ on a neighbourhood of $\mathbb{C}^2 \backslash V$ and $\sup g' = m$. Let then

$$\tilde{g}(x) = \begin{cases} m & \text{if} \quad x \in \mathbb{C}^2 \backslash V \\ \max(g^*(F(x), g'(x)) & \text{if} \quad x \in V. \end{cases}$$

It is clear that $\tilde{g}$ is a new defining function for K' satisfying $L(\tilde{g}) = 0$ in $\mathbb{C}^2$, $\tilde{g}^{-1}(0) = K'$ and $u(x, 0) \leqslant \tilde{g}(x)$ for $x \in V$. (Since F is proper, $g \circ F = m$ near bV.)

Let now $U \in C^0(\mathbb{C}^2 \times [0, +\infty))$ be the solutions of the parabolic problem

$$\begin{cases} U_s = L(U) & \text{in} \quad \mathbb{C}^2 \times (0, +\infty) \\ U = \tilde{g} & \text{on} \quad \mathbb{C}^2 \times \{0\}. \end{cases} \tag{2.10}$$

Let $T > t^*$ where t^* is the extinction time for K'. Consider the open set $H = V \times (0, T)$. Observe that $U = m$ on $bH \backslash V \times \{0\}$ and $U(\cdot, 0) = \tilde{g}$, while $u \leqslant m$ in H and $u(z, 0) \leqslant \tilde{g}(x) \leqslant U(x, 0)$ for $(x, 0) \in V \times \{0\}$. Thus the comparison principle implies that

$$0 \leqslant u(x, t) \leqslant U(x, t)$$

in H, and so, for $t > 0$

$$\begin{aligned} \mathcal{E}_t^{\mathcal{L}}(K') &= \{x \in V : U(x, t) = 0\} \subseteq \{x \in V : u(x, t) = 0\} \\ &= \{x \in V : v(F(x), ct) = 0\} \subseteq \mathring{K}. \end{aligned}$$

If $x \in S$, $F(x) \notin \mathring{K}$, hence $v(F(x), ct) > 0$ for all $t > 0$, i.e., $x \notin \mathcal{E}_t^{\mathcal{L}}(K')$, $t > 0$. ∎

COROLLARY 2.7. *Let $K', K \subset \mathbb{C}^2$ be Stein compacta with Stein neighbourhoods V and W respectively and $F: V \to W$ be a locally biholomorphic map. Assume that $F^{-1}(K) = K'$, $F(K') = K$. Then*

(a) *if the evolution of K is strictly contracting, so is the evolution of K',*

(b) *if K disappears instantaneously, so does K'.*

Proof. Observe first, that we can shrink V and W, if necessary, so that $F: V \to W$ is a proper map.

(a) Clearly $F^{-1}(\mathring{K})$ is open, so

$$F^{-1}(\mathring{K}) \subseteq \mathring{K}' \tag{2.11}$$

and

$$F^{-1}(bK) \cup F^{-1}(\mathring{K}) = K'. \tag{2.12}$$

Since F is locally biholomorphic, $F^{-1}(bK)$ is nowhere dense in $\mathbb{C}^2$. Suppose $x' \in F^{-1}(bK) \cap \mathring{K}' \neq \emptyset$. Then $F(x') \in bK \cap F(\mathring{K}')$. Since $F(\mathring{K}')$ is open and contained in $\mathring{K}$, it is a contradiction. Thus $F^{-1}(bK) \subseteq bK'$. The relations (2.11) and (2.12) imply that $bK' = F^{-1}(bK)$, $\mathring{K}' = F^{-1}(\mathring{K})$, and $F(bK') = bK$.

Apply now Lemma 2.6 with $S = bK'$.

Since $F(S) \subseteq bK$, and the evolution of K is strictly contracting, we obtain

$$\mathscr{E}_t^{\mathscr{L}}(K') \subseteq K' \backslash S = \mathring{K}', \qquad t > 0,$$

i.e., the evolution of K' is strictly contracting.

(b) If K disappears instantaneously, $\mathring{K} = \emptyset$, and since $\mathring{K}' = F^{-1}(\mathring{K})$, and F is locally biholomorphic, $\mathring{K}' = \emptyset$. The fact that $\mathscr{E}_t^{\mathscr{L}}(K) = \emptyset$, $t > 0$, means that the evolution of K is strictly contracting, and so is, by part (a), the evolution of K', i.e., K' disappears instantaneously. ∎

PROPOSITION 2.8. *Let Q_1, Q_2 be two closed rectangles in $\mathbb{C}$ and $K' = Q_1 \times Q_2$. Then the evolution of K' is strictly contracting.*

Proof. Since $bK' = S_1 \cup \cdots \cup S_8$, where the S_j's are the flat faces of K, it is enough to show that $\mathscr{E}_t^{\mathscr{L}}(K) \cap S' = \emptyset$, $t > 0$, where S' is any of the faces. Since the evolution equation is invariant with respect to translation and unitary transformation of $\mathbb{C}^2$, we can assume, without loss of generality, that

$$K' \subset \mathbb{C} \times \{w \in \mathbb{C} : \operatorname{Re} w \leqslant 0\}$$

$$S' \subset \mathbb{C} \times \{w \in \mathbb{C} : \operatorname{Re} w = 0\}.$$

Let $0 < R,\ a,\ d < +\infty$, $\varepsilon > 0$ be such that

$$K' \subset V := D(0, R) \times \{w \in \mathbb{C} : a < \operatorname{Re} w < \varepsilon,\ |\operatorname{Im} w| < d\}.$$

Choose $\alpha > 0$ such that $\alpha d < \pi$. Then the map F: $V \to D(0, R) \times D(0, e^{\varepsilon})$ given by $F(z, w) = (z, e^{\alpha w})$ is biholomorphic (onto $F(V)$). Observe that $F(K') \subseteq K = \bar{D}(0, R) \times \bar{D}(0, e^{\varepsilon})$ and

$$F(S') \subseteq S := \{(z, e^{i\theta}) : |z| \leqslant R\}.$$

By Lemma 1.3, $\mathscr{E}_t^{\mathscr{L}}(K) \subset K \backslash S$, $t > 0$, and so, by Lemma 2.6, $\mathscr{E}_t^{\mathscr{L}}(K') \subset K' \backslash S'$. ∎

COROLLARY 2.9. *Let $\Omega \subset \mathbb{C}^2$ be an arbitrary bounded convex domain. Then the evolution of $\bar{\Omega}$ is strictly contracting and the evolution of $b\Omega$ is stationary.*

Proof. It was proven in [ST_3, Remark 4.3] that the second statement follows from the first.

To prove the first, consider an arbitrary boundary point $\zeta \in b\Omega$. Let Π be a real hyperplane supporting $b\Omega$ at ζ. Choose a parallelpiped K' (as in Proposition 2.8) such that

$$\bar{\Omega} \subseteq K', \qquad \mathring{K}' \cap \Pi = \varnothing.$$

Since, by Proposition 2.8, evolution of K' is strictly contracting, we have for $t > 0$

$$\mathscr{E}_t^{\mathscr{L}}(\bar{\Omega}) \subseteq \mathscr{E}_t^{\mathscr{L}}(K') \subset \mathring{K}',$$

so $\zeta \notin \mathscr{E}_t^{\mathscr{L}}(\bar{\Omega})$, $t > 0$. In view of the arbitrariness of $\zeta \in b\Omega$, $\mathscr{E}_t^{\mathscr{L}}(\bar{\Omega}) \subset \Omega$. ∎

COROLLARY 2.10. *An empty interior compact convex set of $\mathbb{C}^2$ disappears instantaneously.*

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